The positions of the other places mentioned in this paper are given above.

Professor Celoria's extracts from mediæval records represent a vast amount of valuable labour. It is advisable, however, to warn readers that his figures for the 1239 eclipse must not be taken from his original paper, but from his own revised figures in his second paper.

The Irregular Movement of the Earth's Axis of Rotation: a Contribution towards the Analysis of its Causes. By Professor J. Larmor, F.R.S., and Major E. H. Hills, C.M.G.*

Much material, defining with increasing accuracy the irregular wanderings of the Earth's axis of rotation, has now been accumulating for a long series of years. The attempt to decompose the movement into regular harmonic components excited interest some six or eight years ago. Since that time the more systematic data obtained and analysed by the International Organisation have given greater precision to the path of the Pole; and, while a definite astronomical discussion must rest with the experts, curiosity as to the general physical causes of the phenomenon is legitimate. It has long been recognised that displacement of material on the Earth's surface due to meteorological changes (melting of polar ice, long-period barometric fluctuations, etc.) must be a prominent agent, and may indeed be taken to be the main one; while Newcomb has pointed out that the free Eulerian oscillatory period must be very different, for a nearly spherically balanced Earth, if it is elastically deformable under the centrifugal force, from what it would be if it were rigid, thus accounting for the unexpected value of the Chandler period.

It is shown below that, without making any hypothesis except the natural one, that this free precessional period is fixed in duration and determines the average duration of the revolutions of the Pole of rotation, it is easy by a graphical process to deduce from the path of the pole a map of the varying torque which must be acting in order to produce that path, and thence to infer as to the character of the displacements of terrestrial material that must be taking place in order to originate that torque on the Earth as a whole. It has seemed worth while to carry this out in a preliminary way. It has also been thought worth while to set down various dynamical considerations which may prove useful in a systematic analysis of the observational results.

Let ω_1, ω_2, Ω be the component angular velocities of the Earth referred to axes moving with itself, the latter being around the axis of figure. Thus $\omega_1/\Omega, \omega_2/\Omega$ are the angular co-ordinates of the pole of rotation measured on the Earth's surface, and represent

* Read in part at the British Association, August 1906.

directly the change of latitude; they are different from the absolute co-ordinates of the pole on the celestial sphere, in the order of the ratio $C/(C-A)$, as appears from the Poinsot geometrical representation of the free precessional motion, C and A being the *effective* polar and equatorial moments of inertia.* The dynamical equations of the Earth's free precession are, with sufficient accuracy, in terms of the angular momentum (h_1, h_2, H),

$$\frac{dh_1}{dt} - h_2\Omega + H\omega_2 = 0$$

$$\frac{dh_2}{dt} - H\omega_1 + h_1\Omega = 0,$$

where, D, E, F being products of inertia,

$$h_1 = A\omega_1 - F\omega_2 - E\Omega, \quad h_2 = B\omega_2 - D\Omega - F\omega_1, \quad H = C\Omega - D\omega_2 - E\omega_1$$

Thus
$$A\dot{\omega}_1 - (C-A)\Omega\omega_2 = -L'$$
$$A\dot{\omega}_2 + (C-A)\Omega\omega_1 = -M',$$

where Ω may be taken constant, because its variation would multiply in these equations two factors, each small of the first order; and where L', M' include, in addition to the kinetic forcives due to the location of mobile material attached to the earth, the terms

$$\dot{A}\omega_1 - \dot{F}\omega_2 - \dot{E}\Omega, \quad \dot{B}\omega_2 - \dot{D}\Omega - \dot{F}\omega_1$$

Here the terms on the left would be rates of change of angular momentum if the configuration remained constant. The terms on the right include the reversed rates of change of the same, taking the angular velocity constant but the configuration altering. To determine these latter terms, as arising from the reaction of mobile terrestrial material, we can consider them directly as representing the centrifugal forces of this loose material, together with the reversed gradient of the angular momentum, due to the Earth's rotation, of new negative material in the original position and of new positive material in the altered position.

The difference between this procedure and the usual one adopted by previous writers † is that the investigation of the slight

* *Cf.* Routh, *Dynamics*, ii. §§ 180–2, 533. On account of the largeness of this factor $C/(C-A)$ the axis of rotation is practically fixed *in space*, except in so far as the extraneous attractions of the Sun and Moon operate in causing forced precession, so that the effect here in question is simply a change of latitude. *Cf.* Maxwell ["On a Dynamical Top . . . . with some Suggestions as to the Earth's Motion," *Trans. R. S. Edin.*, 1857, *Scientific Papers* i. (pp. 259–61)], who appears to have been the first to apply the Eulerian theory precisely to the Earth, and to examine the Greenwich observations in search of a 306-day period in latitude. According to Professor Newcomb, *Monthly Notices*, lii., 1892, pp. 336–41, searches for terms in the latitude, of this period, instituted at Pulkowa by Peters, and at Washington, in 1862–7, had already yielded negative results. In Newcomb's paper, which pointed out and estimated roughly the increase of free period produced by elastic yielding, the influence of irregular meteorological displacements was also considered.

† For an account with references, *cf.* Routh, *Dynamics*, ii. §§ 24, 533–5.

changes of position of the Earth's principal axes of inertia arising from displacement of material is evaded, by considering an unchanging Earth, with effective moments of inertia (A, A, C), which is subject to force arising from the kinetic reaction exerted on it by this additional and independent material, moving over it and at the same time maintained by it in diurnal rotation.

The effect of centrifugal force, in flattening elastically the terrestrial spheroid, simply modifies its effective or dynamical moments of inertia according to the principle arrived at in a previous discussion,* viz. that the moments of inertia which would exist in the absence of diurnal rotation, but on the assumption that the Earth's form when centrifugal force is thus removed is determined by a linear law of elasticity, are to be employed in dynamical investigations which take account of the elastic yielding of the Earth. There would be consequently an increase in the free precessional period (actually as observed it is from 306 to about 428 days) in the manner first pointed out by Newcomb.

Refer now the problem thus formulated to axes of ω_1 and ω_2 rotating with angular velocity $\Omega(C-A)/A$, that of the undisturbed

* "On the Earth's Free Eulerian Precession," *Proc. Camb. Phil. Soc.*, May 25, 1896, p. 186.

The argument there employed, briefly stated in more analytic form, is as follows. Let A′, B′, C′ be the principal moments of inertia of the Earth when unstrained by centrifugal force; and let I be the change of moment of inertia round its own axis, due to the equatorial protuberance raised by that force. This axis is in the direction of the resultant angular velocity $(\omega_1, \omega_2, \omega_3)$; and it is implied that it is very near to the principal axis of greatest moment C′, so that $\omega_3(=\Omega)$ is practically constant and great compared with ω_1 and ω_2. It is involved also in this restriction that I is a constant up to the first power of ω_1/Ω or ω_2/Ω. Referred to the principal axes, the total component angular momenta are

$$h_1 = A'\omega_1 + I\omega_1,\ h_2 = B'\omega_2 + I\omega_2,\ h_3 = C'\omega_3 + I\omega_3$$

The equations of motion referred to the rotating axes are of the well-known vector type

$$\dot{h}_1 - h_2\omega_3 + h_3\omega_2 = L$$

When A and B are equal, the third of them is

$$\frac{d}{dt}(C\omega_3) = N$$

where C is the effective moment of inertia $C'+I$: when N is null ω_3 is thus constant, say Ω, up to the first order. The other two equations are

$$\frac{d}{dt}\cdot(A'+I)\omega_1 + (C'-B')\Omega\omega = L_2$$

$$\frac{d}{dt}\cdot(B'+I)\omega_2 - (C'-A')\Omega\omega_1 = M,$$

which in the case of approximate symmetry involve a free period $2\pi(A'+I)/(C'-A')\Omega$, and similarly in the general case, thus depending only on A′, B′, C′ when I is small.

The present procedure absorbs the effect of this regular change of form due to strain into modified moments of inertia, while it sets out the effect of erratic displacement of (additional) material on the rotating Earth as a kinetic forcive.

free Eulerian precession, viz., about 428 days. The equations of movement assume the form

$$A\dot{\omega}_1 = -L', \qquad A\dot{\omega}_2 = -M',$$

the same as if the axes were fixed and there were no diurnal rotation; that is, the polar axis moves in the earth, relative to these rotating axes of co-ordinates, along the direction of the reversed resultant of the torques L′ and M′.

It seems useful, therefore, to plot the course of the Pole relative to co-ordinate axes rotating with the mean Chandler period, marking, at intervals along the curve, both the time, and the longitude of one of the revolving axes of co-ordinates at that time; for the velocity along the curve at any instant will then give the direction and magnitude of that part of the rate of change of (L′, M′), the transverse component of the centrifugal torque of the loose material and the time-gradient of the angular momentum arising from transport of this material, when the velocity of rotation is imagined unaltered. Such change can therefore be partially located, as *infra*; if it is mainly due to displacements of surface-material, of thermal or meteorological type, it should show seasonal recurrences, and may prove to be in part due to slight change in oceanic or barometric levels.

Aggregate rough estimates of mere order of magnitude are easiest made directly, without use of these rotating axes. Thus a surface depression of 1 foot over a square mile, extending down in gradually diminishing amount to 30 miles, would involve an effective displacement of a layer 1 foot thick through 15 miles downward. In latitude 45°, where the effect in this respect would be greatest, this displacement would change the resultant transverse component of angular momentum of the earth by $4000\,\Omega\frac{2\frac{3}{4}\cdot 15}{5280}\cos^2 45°$, the whole angular momentum of the earth being in the same units $\Omega.\ 5\frac{1}{2}.\ \frac{4}{3}\pi(4000)^3.\ \frac{2}{5}(4000)^2$; this takes the density of surface material to be $2\frac{3}{4}$ and that of the whole Earth $5\frac{1}{2}$. The polar axis would thereby be displaced through an angle equal in absolute measure to the ratio of these quantities; in seconds of arc it would be about 3.10^{-13}. Thus local displacements by earthquakes can have no sensible direct effect on motion of the Pole. But more important is the centrifugal effect representable by displacement of the axis of inertia of the Earth, round which the free precession of the polar axis is taking place. This angular displacement is $h/(C-A)$, when h is the product of inertia thus introduced: this is of $C/(C-A)$ times (about 300 times) the order of the direct effect on the Pole of rotation, so that in the present way of viewing the matter the changing origin of precession is practically everything.* It has been found, however, by Milne (Bakerian Lecture, *Roy. Soc. Proc.*, 1906), that, for the

* Sir G. H. Darwin has recently expressed the opinion that on the whole the effect of earthquakes may be to bring the axis of rotation nearer to the axis of figure, and thus damp the polar movement.

small range of time (two years) then investigated by him, sharp curvatures in the polar movement appeared to be on the whole concomitant with earthquakes; the latter may be promoted perhaps by the changes of superficial or internal loading along meridians, that are the main cause of the irregular motion of the Pole and are greatest when the curvature of its path is sharpest. The procedure indicated in this note would locate to some extent this displacement or change of loading, and thus test that theory.*

The effect of transfer of water from the Poles towards mean latitudes, arising from melting of Arctic ice, may be estimated either by considering an added layer, of thickness positive or negative according to the locality, and of null aggregate amount spread over the whole ocean, or by estimating directly as above the change of angular momentum involved in the displacement of each portion of the material. A displacement of the Pole of rotation in the Earth in a given direction, when referred to the rotating axes as above, would imply alteration of intrinsic angular momentum of surface load in the neighbourhood of the meridian circle containing that direction, which would be a defect in the northern quadrant in front of that direction or an increase in the northern quadrant behind it, and *vice versa* for the southern quadrants. Thus water rapidly moved from the Poles, where it has little angular momentum, so as to cover to a depth of 1 foot a region 4000 miles square, in middle latitudes, would displace the Pole of rotation in the Earth by something of the order of 2 seconds of arc; for it would involve a new transverse angular momentum $\Omega \cos^2 45^\circ\ 4000^5/5280$ in the same units as above. It is readily seen that the principal axis of inertia, about which the free precession would continue, would be displaced in the opposite direction through an angle of the same order.

In reducing the International Observations of change of latitude at the selected observatories extending round the Earth, it has been found necessary to include a change common to all longitudes, which at first sight could only arise from change of form of the spheroid which represents the terrestrial sea-level. This might be in part due to gravitational influence of the displaced material; yet the removal by melting of 10 feet of ice over a polar area 500 miles in diameter would produce a change of attraction which could not, in middle latitudes, raise the Pole by more than

* The case here considered is 15 cubic miles of material displaced vertically 1 foot. Professor Milne informs us that the result of an actual earthquake might be 10,000,000 cubic miles displaced vertically or horizontally through 10 feet. This would multiply the figure in the text by 7.10^6, thus giving 2.10^{-6} seconds of arc. After this sudden shift of the axis of rotation in the Earth, the free precession would continue, but it would be around a new principal axis of inertia displaced from the original one by a quantity of the same small order of magnitude multiplied by $C/(C-A)$, that is by 300, giving a result of the order 10^{-3} seconds.

Sir G. H. Darwin has estimated (*Phil. Trans.*, 1876) that $\frac{1}{100}$ of the area of Africa rising or falling *in situ* through 10 feet would produce 0·2 seconds of change, the rising or sinking being presumably taken as not merely superficial.

1/1000 of the order of magnitude required. As the effect seems to be real, some indirect, perhaps seasonal instrumental, cause must apparently be sought for.

In trying, as here, to separate out the meteorological displacements of the Pole from the true free precession which would in their absence be a regular circular motion, by referring the whole to rotating axes, the essential point is to assign as correctly as possible the period of this precession, for that determines the velocity of rotation to be given to the axes of co-ordinates. Elastic yielding of the Earth will prolong the period beyond the 306 days that would belong to a rigid solid. Hough has shown * that an average modulus of rigidity of a solid Earth even so great as that of steel would involve the prolongation of the period to the above value, 428 days, which represents the periodicity of the observed path of the Pole. Now whatever be the cause, elastic or fluid displacement, or both, that thus alters the effective dynamical moments of inertia of the Earth, it may be presumed that it alters them to a constant extent over a fairly long period of time. Thus the true free, or Eulerian, precession would maintain a rotation of the Pole fairly constant, while the meteorological disturbance superposed on it would in the long-run have no rotational quality one way round or the other. In applying the method of analysis of the complex motion that is here proposed, the angular velocity of true precession may therefore be obtained by taking the mean of the observed times of revolution of the Pole, as Chandler originally pointed out.

The Eulerian principal axes fixed in the Earth, or still better the axes rotating as specified above, are appropriate to the analysis of the effects of moments or torques, which arise from change of loading and are therefore themselves revolving on the whole with the Earth. On the other hand, torques which only slowly change in direction in space, such as those arising from the attraction of the Sun and Moon on the protuberant parts of the oblate terrestrial spheroid, are most amenable to dynamical analysis when referred to fixed axes. They produce mainly the ordinary forced astronomical precessions and nutations, on which the varying elastic yielding of the Earth has no sensible kinetic influence; while the intensity of the torque of attraction depends on the instantaneous geometrical value of $C - A$, as determined by distribution alone. The distribution of mass thus governs the solar and lunar precessions.† These forced precessions depend on internal fluidity,

* *Phil. Trans.*, 1895. It is (as above) the near approach to sphericity that makes so slight a yielding to the changes of centrifugal and other stresses effective in this manner.

† There is one case, however, in which, as the equations of motion given above show, a small extraneous forcive would cause a wandering of the Pole in the Earth itself, much greater than its change of direction in space, *i.e.* than the astronomical precession thereby caused, that, namely, of a forcive having a period in longitude relative to the Earth's rotation, of the order of the free period of 428 days. Such a term in the forcive would cause a forced

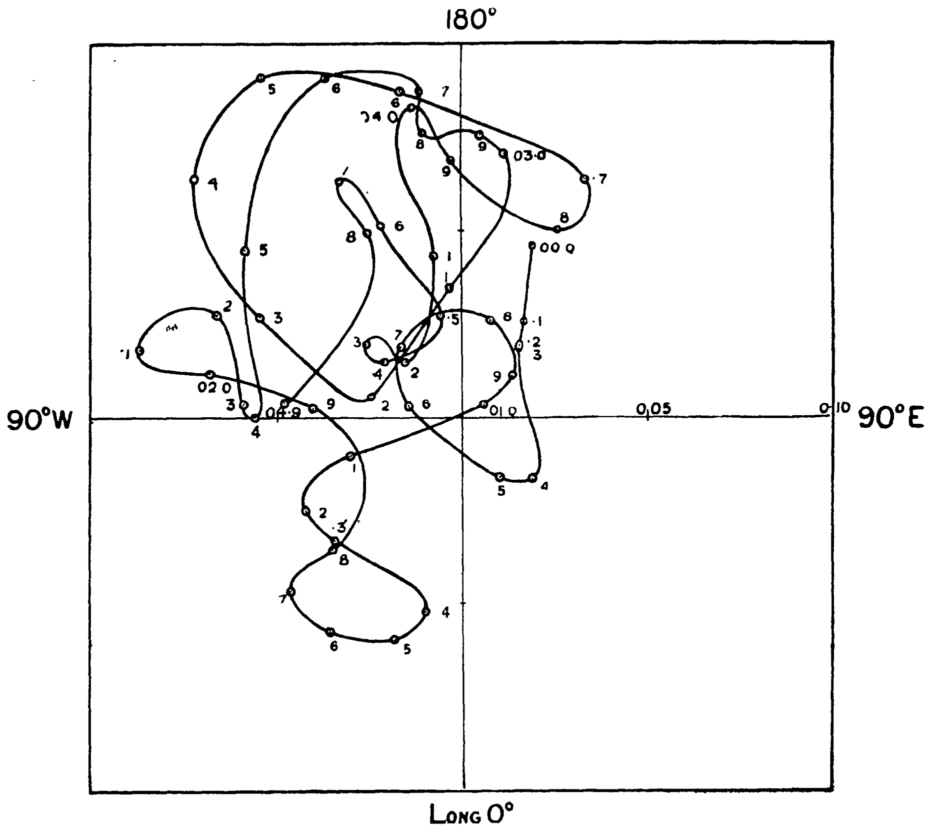
etc., only in so far as it modifies the effective inertia-moment C or A; while, on the other hand, the free precession depends on the effective or kinetic value of the small difference $(C-A)/C$.

The tides would be about the same at antipodal points if contours of the land were symmetrical, and the two opposite tidal protuberances would be additive in their effect on the free precessional moment. At first sight, it might appear that the estimate given above for a local overflow of a foot of water from the Polar regions would involve that, even apart from the irregularity of form of the oceans, the features of the various tidal components travelling round the Earth must be reproduced to some extent in an exact diagram of the torque, owing to the tidal flow demanding alteration of the angular momentum of the water relative to the Earth's rotation, and to its centrifugal force. But in so far as it is the Earth that turns round under the nearly stationary tide, the direct dynamical effects of the tidal movements are very slight and belong to the astronomical class, and are in fact merged in the lunar and solar nutations. But if the Earth's surface were divided by meridional barriers, so that the tidal flow would be in the main north and south instead of around the Earth, we might expect a tidal aberration in the latitude of amount not entirely insensible.

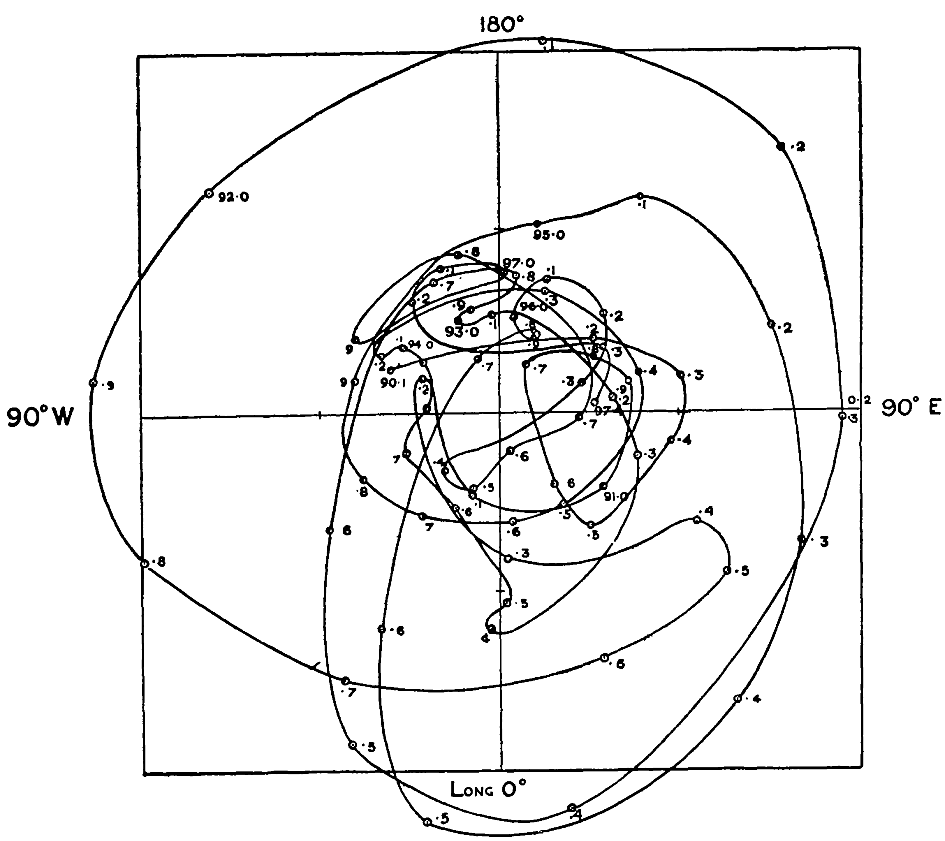
The magnitude of this tidal torque would then compare even with that of the precessional couple of the Moon's attraction on the protuberant parts of the terrestrial spheroid. The latter is $\frac{2}{3}\,Mr^{-3}\,(C-A)\sin 2\delta$, so that its amplitude is about $\frac{3}{2}(0''\cdot 1\; g/a)\; 10^{-2}\,C$ while the earth's angular momentum is $C\omega$. If this torque were to rotate with the Earth for six hours it would produce an angular displacement of the Pole of amount $0''\cdot 001\; g.6h/\omega a$, where $\omega^2 a/g = \frac{1}{289}$, that is, of amount $0''\cdot 001 \times 289.\frac{1}{2}\pi$ or $0''\cdot 5$; in contrast with the actual lunar fortnightly nutation of amplitude one or two seconds. It has just been seen that a partial tidal overflow from the poles covering $(4000)^2$ square miles to the depth of a foot, and carried on with the Earth's rotation, could in an extreme case account for a displacement of the Pole as much as $2''$: and the antipodal high waters reinforce each other. It is true that if the Earth were covered symmetrically with water, the Sun and Moon travelling in the equator would produce no effect. But the obliquity of their paths and the irregular distribution of the oceans must lead indirectly to nutations of the pole of the short periods of the various tidal components, which are not insensible in the present connection. Thus, for example, a forced nutation of this kind might introduce discrepancies into observations around a parallel of latitude, of character in part systematic owing to the progressive semi-diurnal change of phase, which would be eliminated in smoothing out the observations for each observing station of the International chain of longitude.

Possibilities in this direction are perhaps worth bearing in mind.

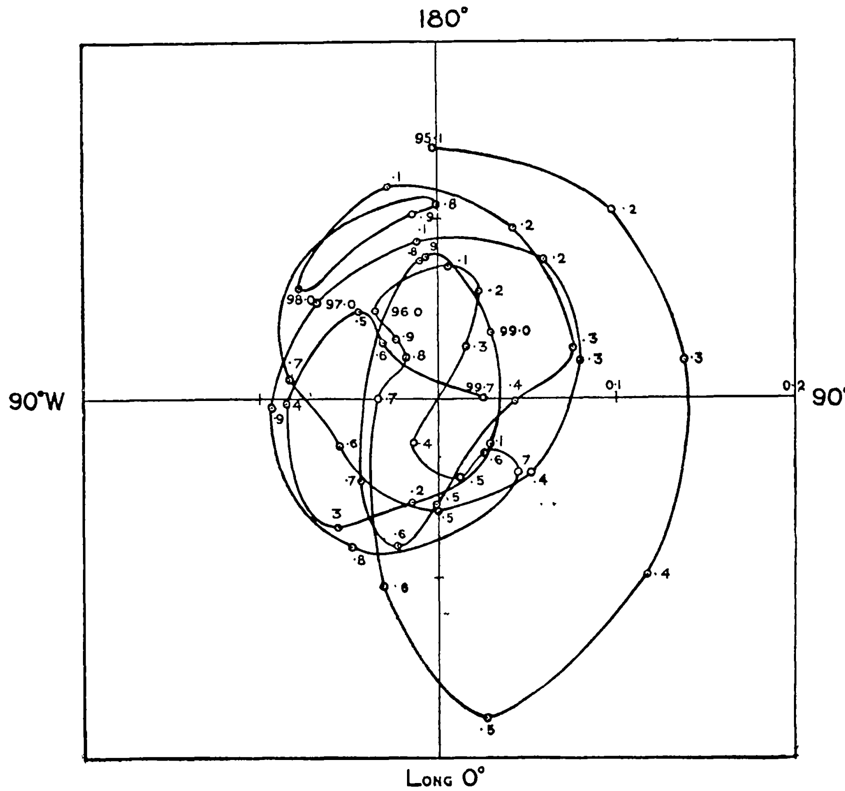
oscillation in latitude of its own period, which would be superposed on the free oscillation of a nearly equal period, thus producing an alternation in its amplitude.



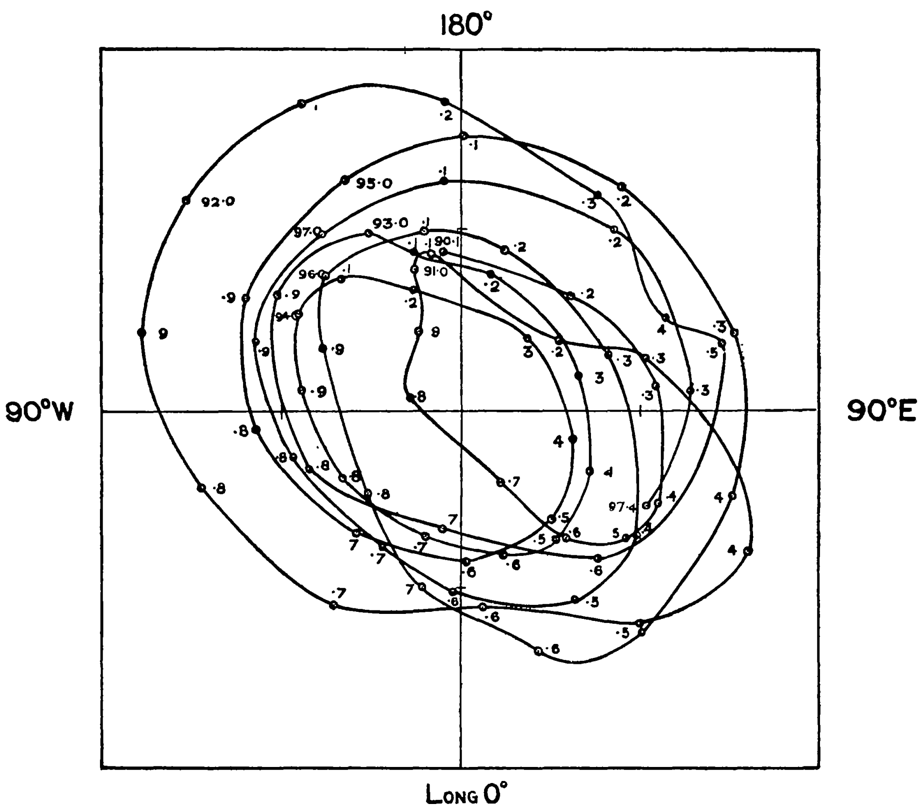
1. ALBRECHT, 1900-1905.



2. ALBRECHT, 1890-1897.



3. CHANDLER, 1895-1899.



4. CHANDLER, 1890-1897.

For example, in addition to the difference of phase at different stations mentioned above, if the same stars were observed at all the stations, a solar tidal nutation might partially simulate a change of latitude with a yearly period, common to all the stations, such as appears in the reduced observations. The deviation of the vertical by the attraction of the tidal water is in all cases very small compared with this deviation of the Pole (*cf.* Thomson and Tait's *Natural Philosophy*, ed. 2), and is in fact here negligible.

The influence of a nutational torque of fairly short period could be laid off by aid of the diagram referred to rotating axes; but even in this case it can be effected rather more easily in the usual astronomical manner, on account of the relatively slow change of its direction in space.

But whether these tidal influences are sensible or not, assuming 428 days to be the period of free precession, we can transform the curve of wandering of the Pole as above to axes of co-ordinates rotating with this period: and the hodograph of this new curve, when referred back again to axes connected with the Earth, will represent the distribution in direction and time of the torque arising from displacement of terrestrial material, which is continually modifying the motion of the Pole. From the point of view of geophysics, this curve would appear to be worth setting out, and might be expected to show seasonal recurrences.

The annexed diagram (1) gives, from the path of the Pole since 1900, as officially published by Professor Albrecht in the *Ast. Nachr.*,* the torque which must have been in action to cause that motion. It will be observed that the expected annual periodicity does not appear, but that the direction of the axis of the torque points preponderantly towards the side of the Pacific Ocean, an extraneous feature which we shall show how to eliminate later (p. 31).

Then follow, but based on more imperfect data, the torque-diagram (2) derived from the Albrecht diagram of Polar wandering for the period 1890–97. The other two are derived from the Chandler diagrams for the periods 1895–99 and 1890–97. It will be noticed that for the first one and a half years of the overlapping period 1895–97 the torque-diagram is much the same in both, but not for the later part. The appearance of all these diagrams is very different from that of the first, *e.g.* the one-sided bias in the torque does not appear.

The unit (0·1) marked on the axes of the diagrams represents the torque that would shift the Pole at the rate of 0″·05 in one-tenth of a year; the dates are marked along the curves in decimals of a year.

The analysis of Chandler made the motion of the Pole consist of his circular precession of 428 days period, with an additional elliptic

* The most convenient way to use these diagrams is to place them on a terrestrial globe, with the origin at the Pole. The Albrecht and Chandler path-diagrams from which these force-diagrams have been constructed may be found in *Bericht über den Stand . . . der Breitvariation*, and *Ast. Journal*, vol. xix., respectively.

motion about the centre, and of yearly period, superposed on it. The component torques that would originate such an addition to a free precession of 428 days, being proportional to $\dot{\omega}_1 - n\omega_2$ and $\dot{\omega}_2 + n\omega_1$, would have an annual elliptic periodicity, in rough agreement with this diagram. For, referring to the axis of the ellipse, this theoretical motion would be

$$\omega_1 = \mathrm{A} \cos (nt + \alpha) + a \cos pt$$
$$\omega_2 = \mathrm{A} \sin (nt + \alpha) + b \sin pt,$$

so that

$$\dot{\omega}_1 - n\omega_2 = -(ap + bn) \sin pt$$
$$\dot{\omega}_2 + n\omega_1 = (an + bp) \cos pt,$$

in which the amplitude A of the free precession is not involved.

We have now to inquire into the kind of information that these diagrams can convey. On our plan of analysis, on the basis of a definite elastic Earth on which additional matter can be displaced, the forcive necessary to supply new angular momentum to material that has come into a position of greater diurnal velocity has to be supplied by this earth, while it has also to sustain the centrifugal force of this material. The other ways in which the mobile material reacts on the motion of the Earth on which it is superposed are negligible in comparison with these two.*

For a mass m in co-latitude θ the centrifugal torque is $m\Omega^2 r^2 \sin\theta \cos\theta$ around an axis at right angles to the meridian of m. The aggregate torque corresponding to its angular momentum in its present position is $m\Omega r^2 \sin\theta\cos\theta$, as regards the equatorial component which is in the meridian of m. The former operates as a whole as a forcive, but only the time-gradient of the latter thus acts, which is $2m\Omega rv \cos 2\theta$, where v is the velocity of m along the meridian. The former preponderates, in the ratio $\frac{1}{2}\Omega r$ to v, therefore usually very much so, as $\frac{1}{2}\Omega r$ is of the order of 300 miles per hour. On the other hand, the centrifugal force of a new steady local load merely makes the steady precession occur about a new axis of inertia, thus is not progressive or cumulative. This is readily verified by reversing the graphical procedure; a steady torque, represented by the end of a radius vector, becomes represented by an arc of a circle when referred to axes rotating with the free precessional velocity, and this corresponds to a velocity of circular precession of the Pole while it lasts.

This double mode of action of transported material, through centrifugal force and through change of momentum of diurnal motion, renders interpretation of the torque-diagram to some extent indefinite.

But, neglecting the effect of change of intrinsic angular momentum, which may be as much as one-tenth of the whole, the torque will be due to the centrifugal force of the distribution

* This is readily seen by the procedure of the note, p. 24, if we introduce the exact formula $h_1 = \mathrm{A}'\omega_1 - \mathrm{E}\omega_3 - \mathrm{F}\omega_2 + \mathrm{I}\omega_1$, where for an additional mass m at xyz, $\mathrm{F} = mxy$, $\mathrm{E} = mxz$; we are in fact merely neglecting $\Omega\omega_1$ and $\Omega\omega_2$ compared with Ω^2.

of the mobile load at each instant, and will thus indicate the general features of that distribution; while the rate of change of the torque, *i.e.* the velocity in this torque-diagram, will give an indication of the movement of the load. The radius vector O P of the torque-diagram at any instant will imply a proportional accumulation of materials in middle latitudes on the meridian at right angles to O P, so that antipodal accumulations reinforce, but adjacent ones, north and south of the equator, counteract each other. The marked tendency of the torque-diagram for the period 1900–05 towards the side of the Pacific Ocean might thus be due to simultaneous accumulation of load not on the side of the Pacific, but in the neighbourhood of the perpendicular meridian. There is, however, a possible alternative to be kept in view, as follows.

It has been suggested to us by Professor Turner that the position of the origin to which the curve of wandering of the Pole of rotation is referred, is subject to considerable uncertainty. The observations and their reduction do not, however, seem to be at fault specially in this direction; unless the unexplained constant (Kimura) term may be taken to indicate a radius of uncertainty due to seasonal instrumental changes. This term, which recent discussion has confined to a smaller amplitude and to an annual period, was referred roughly to a displacement (mainly N and S) of the Earth's centre of gravity: we have verified above, however, that no likely meridional transfer due to seasonal change of temperature could produce an effect so great.

There does not, in fact, seem to be any ground, apart from mere uncertainties, for taking the origin to which the wanderings of the Pole of rotation are referred to be other than a fixed point on the Earth. But, on the other hand, this fixed point may not be the Pole of inertia of the solid Earth. We can, however, make it so, by separating from the solid Earth a thin superficial sheet, and counting this with the mobile material. The centrifugal forcive due to this sheet will then constitute a torque invariable with respect to the Earth: and we have merely to subtract this from the torque-diagram referred to the Earth in order to obtain the torque due to the loose material alone. This subtraction of a constant vector term amounts simply to a change of origin. Thus on the final torque-diagram the origin is uncertain, and would naturally be placed in as central a position as possible.

The process here carried out graphically may be compared with the procedure by successive steps as employed by Newcomb, in which free precession occurs for an infinitesimal time δt round an axis of inertia O supposed fixed in the Earth, then O is moved on to O_1 as the result of the change of the mobile load during that time, then free precession takes place round O_1 for a time δt_1, then O_1 is moved on to O_2, etc. Our method has virtually amounted to the elimination of the free precessional motion, with its constant angular velocity, thus leaving the causes which displace the pole of inertia on the Earth's surface open to inspection—the constrained

shift of the axis of rotation in space being neglected for the nearly spherical Earth, as *infra*.

How far the results may throw light on their causes depends largely on a comparison of the diagrams with the displacements of matter on the Earth's surface that are known to meteorology and oceanography. It may be that not much certain information may yet be derivable; but, considering the long time that observations of the wandering of the Pole have been accumulating, it can hardly be said that it is too soon to prepare for their preliminary discussion from the geophysical point of view.

The mode of reduction on which this paper is founded gives a force diagram which exhibits the torque sustained by the rest of the Earth owing to the displacement in and over it of the movable masses treated as independent bodies. It remains valid, however rapid the free precession may be. In the case of the actual earth the latter is slow, being $C/(C-A)$ sidereal days, in which the difference of moments of inertia has its effective or dynamical value, thus lengthening the free period from 306 to 428 solar days. In this case various features assume simple forms, as appears in the Poinsot representation by a rolling ellipsoid. When this ellipsoid is very nearly spherical, the axis of rotation, drawn from its fixed centre to the point of contact with the plane on which it rolls, is at a small inclination to the invariable direction of resultant momentum (which is normal to that plane) compared with its inclination to the axis of inertia; thus the axis of rotation is practically fixed in direction in space—except as regards the superposed luni-solar precession. When the ellipsoid of inertia is nearly of revolution, as in the case of the Earth, the pole of inertia thus revolves in space with the uniform free precessional velocity, around the fixed direction of the pole of rotation, while at the same time it is undergoing such shifts as the redistribution of material geometrically requires. Our procedure in the above has been to eliminate the uniform precession, and the residue is a graphical representation of the irregular shifts, or rather of the torques which produce them.

The discussion of the question whether in past geological history the pole of rotation has wandered extensively in the Earth seems also capable of being based on simple graphic representation. We shall assume, as before, that the dynamics of the Earth's rotation are based at each instant on a simple kinetic energy T given by

$$2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2,$$

where A, B, C are effective moments of inertia; it follows that the angular momentum (L, M, N) is given by the formula

$$(L, M, N) = (A\omega_1, B\omega_2, C\omega_3)$$

During the history of the Earth (L, M, N), resultant G, must have remained constant, while T will probably have diminished through

frictional agency. Abstraction is here made of the solar and lunar forced precessions, which compensate the torque of extraneous attractions without affecting the position of the axis of rotation in the Earth. The free motion is represented, after Poinsot, by the angular motion of a momental ellipsoid, say

$$Ax^2 + By^2 + Cz^2 = K$$

The direction of the axis of instantaneous rotation intersects its surface at the point (xyz) such that

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}, \text{ therefore} = \left(\frac{K}{2T}\right)^{\frac{1}{2}}$$

The distance p of the tangent plane at this point (xyz) from the centre is given by

$$\frac{1}{p^2} = \frac{A^2}{K^2}x^2 + \frac{B^2}{K^2}y^2 + \frac{C^2}{K^2}z^2$$
$$= \frac{G^2}{2KT}$$

Thus if $K^{-1} = 2T$, we have $p = G^{-1} =$ constant; this tangent plane is then fixed as regards distance from the centre, as well as regards its direction, which is perpendicular to the invariable momentum (L, M, N). Thus it is entirely fixed.

Thus the free precessional motion of the slowly changing Earth is represented by the varying ellipsoid

$$Ax^2 + By^2 + Cz^2 = (2T)^{-1}$$

rolling with centre fixed, so as to keep in contact with this fixed plane whose distance from the centre is G^{-1}.

If the kinetic energy keeps constant, this is simply Poinsot's representation. The axis of rotation will circulate in the body around the axis of greatest moment of inertia, and in space around the axis of resultant angular momentum. The amplitude of this free precession will be kept small by internal friction, so that the axis of rotation will always be near the principal axis, and can never wander further from its original position than the latter does. It will require a good deal of change of distribution of mass to move this principal axis very far: to move it into the equator the radius of the Earth must shrink to the order of 40 miles along the equator near the new Poles, and expand to about an equal extent near the original Poles.

If the Earth is shrinking uniformly, the moments of inertia vary as l^2, where l represents linear dimensions; thus the angular velocities vary as l^{-2} and the kinetic energy varies as l^{-2}. Thus the dimensions of this rolling ellipsoid remain unaltered. The distance of the plane on which it rolls also keeps fixed. Hence uniform shrinkage without frictional loss of energy would not affect the amplitude of the free precession, or cause the Poles to migrate.

Diminution of the energy of rotation through internal friction, without change of (A, B, C), would increase the dimensions of the rolling ellipsoid (in the proportion of $T^{-\frac{1}{2}}$), and so would make strongly for stability of the axis of rotation, as above remarked. Such increase can only proceed to a limited extent, determined by the ellipsoid just failing to intersect, and so touching the fixed plane at the extremity of its axis.

A discussion of the geological problem of displacement of the polar axis in the Earth must take account of considerations such as these.

The Systematic Motions of the Stars.

By A. S Eddington, B.A., M.Sc.

1. *Introduction*

Of late years astronomers who have investigated the proper motions of stars have been mainly interested in determining the direction of the solar motion. It is usual to assume that, if we consider a sufficient number of stars, their true motions will be at random; on this assumption the direction of the sun's motion has been calculated.

Professor J. C. Kapteyn * has examined the Bradley proper motions to find out whether this assumption is approximately true. He concludes that it is incorrect. Relative to the sun, he finds two "favoured" directions of motion instead of one. Kapteyn suggested that there are two systems or "drifts" of stars; these two drifts are in motion relative to one another. If the whole universe forms one system (or one chaos) we can speak of its motion relative to the sun; but it is more natural, though perhaps misleading, to speak of the sun's motion relative to it. But if there are two systems, we may as well drop the idea of the solar motion altogether, and speak of the motions of the two drifts relative to the sun.

In this paper I have attempted to subject Kapteyn's theory to a quantitative test by examining the Greenwich-Groombridge proper motions. I was led to undertake this by the following considerations:—

(1) The importance and revolutionary character of Kapteyn's discovery render independent confirmation very desirable.

(2) The Greenwich-Groombridge proper motions, recently determined by Dyson and Thackeray, afford new material for this purpose. I had the advantage of access to Dyson and Thackeray's calculations used in their determination of the so-called solar apex.

(3) The Bradley stars are all bright stars. The Groombridge catalogue includes a large proportion of stars between the seventh and ninth magnitudes; it is desirable to find out whether these fall into Kapteyn's two drifts.

* *Brit. Assoc. Report*, 1905.